Homework 8
MTH 829 Complex Analysis

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Proposition 0.1 (Exercise X.10.3)**.**

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$$

Proof. For $r > 0$, let S_r denote the half circle parametrized by $\gamma_r(t) = re^{it}$ with $t \in [0, \pi]$. For $R > 0$ and $\epsilon > 0$, let $\Gamma_{\epsilon,R}$ be the closed curve

$$\Gamma_{\epsilon,R} = [\epsilon, R] \cup S_R \cup [-R, -\epsilon] \cup (-S_\epsilon)$$

We will evaluate the above integral by evaluating

$$\lim_{\epsilon \to 0} \lim_{R \to \infty} \int_{\Gamma_{\epsilon,R}} \frac{1 - e^{2iz}}{z^2} dz = \lim_{\epsilon \to 0} \lim_{R \to \infty} \left(\int_{[\epsilon,R]} + \int_{S_R} + \int_{[-R,\epsilon]} + \int_{-S_\epsilon} \right) = 0$$

This is equal to zero because $\frac{1-e^{2iz}}{z^2}$ is holomorphic away from $z = 0$, and zero is in the exterior of $\Gamma_{\epsilon,R}$. First consider the integral over S_R, which we can rewrite as

$$\int_{S_R} \frac{1 - e^{2iz}}{z^2} dz = \int_0^\pi \frac{1 - e^{2iRe^{it}}}{(Re^{it})^2} (iRe^{it}) dt = i \int_0^\pi \frac{1 - e^{2iRe^{it}}}{Re^{it}} dt$$

We can bound the integrand by

$$\left| \frac{1 - e^{2iRe^{it}}}{Re^{it}} \right| = \frac{|1 - e^{2iRe^{it}}|}{R} \leq \frac{1 + |e^{2iRe^{it}}|}{R}$$

and we can bound $|e^{2iRe^{it}}|$ by

$$\left| e^{2iRe^{it}} \right| = \left| e^{2iR(\cos t + i \sin t)} \right| = \left| e^{-2R \sin t} \right|$$

thus

$$\lim_{R \to \infty} \left| e^{2iRe^{it}} \right| = 0 \implies \lim_{R \to \infty} \left| \frac{1 - e^{2iRe^{it}}}{Re^{it}} \right| = 0 \implies \lim_{R \to \infty} \left| \int_{S_R} \frac{1 - e^{2iz}}{z^2} dz \right| = 0$$

Now consider the integrals on the real axis.

$$\int_{[-R,-\epsilon]} \frac{1-e^{2iz}}{z^2}dz = \int_{-R}^{-\epsilon} \frac{1-e^{2iz}}{z^2}dz = -\int_{-\epsilon}^{-R} \frac{1-e^{2iz}}{z^2}dz = \int_{\epsilon}^{R} \frac{1-e^{-2iz}}{z^2}dz$$

First, note that

$$\sin^2 z = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 = \frac{\left(e^{iz}\right)^2 - 2e^{iz}e^{-iz} + \left(e^{-iz}\right)^2}{-4} = \frac{2-e^{2iz}-e^{-2iz}}{4}$$

This comes in handy, because now we can recognize the integral over the real axis as a multiple of the real integral we want to evaluate.

$$\begin{aligned}\left(\int_{[-R,-\epsilon]} + \int_{[\epsilon,R]}\right) \frac{1-e^{2iz}}{z^2}dz &= \int_{\epsilon}^{R} \frac{1-e^{2iz}}{z^2} + \frac{1-e^{-2iz}}{z^2}dz \\ = \int_{\epsilon}^{R} \frac{2-e^{2iz}-e^{-2iz}}{z^2}dz &= \int_{\epsilon}^{R} \frac{4\sin^2 z}{z^2}dz\end{aligned}$$

Putting this all together,

$$\begin{aligned}\int_0^{\infty} \frac{\sin^2 x}{x^2}dx &= \frac{1}{4}\lim_{\epsilon\to 0}\lim_{R\to\infty}\left(\int_{[-R,-\epsilon]} + \int_{[\epsilon,R]}\right)\frac{1-e^{2iz}}{z^2}dz \\ &= \frac{1}{4}\left(\lim_{\epsilon\to\infty}\lim_{R\to\infty}\int_{\Gamma_{\epsilon,R}} f(z)dz - \lim_{R\to\infty}\int_{S_R} f(z)dz - \lim_{\epsilon\to 0}\int_{-S_\epsilon} f(z)dz\right) \\ &= \frac{1}{4}\lim_{\epsilon\to 0}\int_{S_\epsilon} f(z)dz\end{aligned}$$

So we just need to compute the integral over S_ϵ, which needs to be 2π to get the desired result. First, we need a particular limit. Note that we have the power series

$$1-e^{2iz} = 1 - \sum_{n=0}^{\infty} \frac{-(2iz)^n}{n!} = \sum_{n=1}^{\infty} \frac{-2^n i^n z^n}{n!}$$

so

$$\frac{1-e^{2iz}}{z} = \sum_{n=1}^{\infty} \frac{-2^n i^n z^{n-1}}{n!}$$

thus

$$\lim_{z\to 0} \frac{1-e^{2iz}}{z} = \lim_{z\to 0} \sum_{n=1}^{\infty} \frac{-2^n i^n z^{n-1}}{n!} = -2i$$

Now that we have this limit, we write the integral of S_ϵ as

$$\int_{S_\epsilon} \frac{1-e^{2iz}}{z^2}dz = \int_0^{\pi} \frac{1-e^{2i\epsilon e^{it}}}{(\epsilon e^{it})^2}(i\epsilon e^{it})dt = i\int_0^{\pi} \frac{1-e^{2i\epsilon e^{it}}}{\epsilon e^{it}}dt$$

And now we do some estimation:

$$\left|\int_0^\pi \frac{1-e^{2i\epsilon e^{it}}}{\epsilon e^{it}}dt + 2\pi i\right| = \left|\int_0^\pi \left(\frac{1-e^{2i\epsilon e^{it}}}{\epsilon e^{it}} + 2i\right) dt\right| \leq \int_0^\pi \left|\frac{1-e^{2i\epsilon e^{it}}}{\epsilon e^{it}} + 2i\right| dt$$

Using what we showed above $\left(\lim_{z\to 0} \frac{1-e^{2iz}}{z} = -2i\right)$, there exists $\epsilon > 0$ so that

$$|z| \leq \epsilon \implies \left|\frac{1-e^{2iz}}{z} + 2i\right| < \frac{\epsilon}{\pi}$$

Of course, $|\epsilon e^{it}| \leq \epsilon$, so

$$\left|\frac{1-e^{2i\epsilon e^{it}}}{\epsilon e^{it}} + 2i\right| < \frac{\epsilon}{\pi}$$

Thus

$$\left|\int_0^\pi \frac{1-e^{2i\epsilon e^{it}}}{\epsilon e^{it}}dt + 2\pi i\right| \leq \int_0^\pi \left|\frac{1-e^{2i\epsilon e^{it}}}{\epsilon e^{it}} + 2i\right| dt \leq \frac{\epsilon}{\pi}(\pi) = \epsilon$$

which says that

$$\lim_{\epsilon\to 0} \int_0^\pi \frac{1-e^{2i\epsilon e^{it}}}{\epsilon e^{it}}dt = -2\pi i$$

Rewriting this, we get exactly what we wanted:

$$\lim_{\epsilon\to 0} \int_{S_\epsilon} \frac{1-e^{2iz}}{z^2}dz = \lim_{\epsilon\to 0} i \int_0^\pi \frac{1-e^{2i\epsilon e^{it}}}{\epsilon e^{it}}dt = i(-2\pi i) = 2\pi$$

To recap,

$$\begin{aligned}
\int_0^\infty \frac{\sin^2 x}{x^2}dx &= \frac{1}{4} \lim_{\epsilon\to 0} \lim_{R\to\infty} \left(\int_{[-R,-\epsilon]} + \int_{[\epsilon,R]}\right) \frac{1-e^{2iz}}{z^2}dz \\
&= \frac{1}{4}\left(\lim_{\epsilon\to\infty} \lim_{R\to\infty} \int_{\Gamma_{\epsilon,R}} f(z)dz - \lim_{R\to\infty} \int_{S_R} f(z)dz - \lim_{\epsilon\to 0} \int_{-S_\epsilon} f(z)dz\right) \\
&= \frac{1}{4} \lim_{\epsilon\to 0} \int_{S_\epsilon} f(z)dz = \frac{1}{4}(2\pi) = \frac{\pi}{2}
\end{aligned}$$

□

Proposition 0.2 (Exercise X.10.4)**.** *Let $a \geq 0$ and $b > 0$. Then*

$$\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}dx = \frac{\pi e^{-ab}(ab+1)}{4b^3}$$

Proof. Let $a \geq 0$ and $b > 0$. Define

$$f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$$

Then f is holomorphic except at $\pm ib$, which are poles of order 2, since the denominator factors as $(z-ib)^2(z+ib)^2$. Let S_R be the semicircle of radius R in the upper half plane, parametrized by Re^{it} with $t \in [0, \pi]$, and let Γ_R be the contour $[-R, R] \cup S_R$. The only pole inside Γ_R is ib, so by the residue theorem, for $R > b$, we have

$$\int_{\Gamma_R} f(z)dz = 2\pi i \operatorname{res}_{ib} f$$

We compute the residue as

$$\begin{aligned}\operatorname{res}_{ib} f &= \lim_{z\to ib} \frac{\partial}{\partial z}(z-ib)^2 f(z) = \lim_{z\to ib} \frac{e^{iaz}(iaz-ab-2)}{(z+ib)^3} = \frac{e^{-ab}(iaib-ab-2)}{(2ib)^3} \\ &= \frac{e^{-ab}(-2ab-2)}{-8ib^3} = \frac{2e^{-ab}(ab+1)}{8ib^3} = \frac{-ie^{-ab}(ab+1)}{4b^3}\end{aligned}$$

Thus

$$\int_{\Gamma_R} f(z0dz = 2\pi i \frac{-ie^{-ab}(ab+1)}{4b^3} = \frac{\pi e^{-ab}(ab+1)}{2b^3}$$

Note that along the real axis, we have a nice relationship between the real part of f and the real function we want to integrate.

$$\begin{aligned}\int_{[-R,R]} f(x)dx \int_{-R}^{R} \frac{\cos(ax) + i\sin(ax)}{(x^2+b^2)^2}dx &= \int_{-R}^{R} \frac{\cos(ax)}{(x^2+b^2)^2} + i\int_{-R}^{R} \frac{\sin(ax)}{(x^2+b^2)^2}dx \\ &= \int_{-R}^{R} \frac{\cos(ax)}{(x^2+b^2)^2}dx = 2\int_{0}^{R} \frac{\cos(ax)}{(x^2+b^2)^2}dx\end{aligned}$$

The integral involving $\sin(ax)$ vanishes because $\frac{\sin(ax)}{(x^2+b^2)^2}$ is an odd function, so the integral from $-R$ to zero cancels the integral from zero to R. The last equality follows because $\cos(ax)$ is even, so the integral from $-R$ to zero is equal to the integral from zero to R. Thus

$$\lim_{R\to\infty} \int_0^R \frac{\cos(ax)}{(x^2+b^2)^2}dx = \frac{1}{2}\left(\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz - \lim_{R\to\infty} \int_{S_R} f(z)dz\right)$$

Now we claim that as $R \to \infty$, the contribution from the integral over S_R goes to zero. Taking the absolute value, we get

$$\begin{aligned}\left|\int_{S_R} \frac{e^{iaz}}{(z^2+b^2)^2}dz\right| &\leq \int_{S_R} \frac{|e^{iaz}|}{|(z^2+b^2)^2|}dz = \int_0^\pi \frac{\left|e^{iaRe^{it}}\right|}{|(Re^{it})^2+b^2)^2|}dz = \int_0^\pi \frac{\left|e^{iaR(\cos t + i\sin t)}\right|}{|R^2e^{2it}+b^2|^2}dz \\ &= \int_0^\pi \frac{e^{-aR\sin t}}{|R^2e^{2it}+b^2|^2}dz \leq \pi R \frac{e^{-aR\sin t}}{|R^2e^{2it}+b^2|^2}\end{aligned}$$

which we can see clearly goes to zero as $R \to \infty$. Thus

$$\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}dx = \frac{1}{2}\lim_{R\to\infty}\int_{\Gamma_R} f(z)dz = \frac{1}{2}\left(\frac{\pi e^{-ab}(ab+1)}{2b^3}\right) = \frac{\pi e^{-ab}(ab+1)}{4b^3}$$

□

Proposition 0.3 (Exercise X.10.6). *For $a > 1$, we have*

$$\int_0^{2\pi} \frac{\cos t}{a - \cos t} dt = 2\pi \left(\frac{a}{\sqrt{a^2 - 1}} - 1 \right)$$

Proof. Let $a > 1$ and define $\alpha = a + \sqrt{a^2 - 1}$ and $\beta = a - \sqrt{a^2 - 1}$. Then we can rewrite our integrand as

$$\frac{\cos t}{a - \cos t} = \frac{\frac{1}{2}\left(e^{it} + e^{-it}\right)}{a - \frac{1}{2}\left(e^{it} + e^{-it}\right)} \left(\frac{2e^{it}}{2e^{it}} \right) = \frac{\left(e^{it}\right)^2 + 1}{-\left(e^{it}\right)^2 + 2ae^{it} - 1} = \frac{\left(e^{it}\right)^2 + 1}{-\left(e^{it} - \alpha\right)\left(e^{it} - \beta\right)}$$

This motivates us to define a function

$$f(z) = \frac{z^2 + 1}{-iz(z - \alpha)(z - \beta)}$$

which is holomorphic except at simple poles z, α, β. Let γ be the unit circle, parametrized by $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. Then

$$\begin{aligned} \int_\gamma f(z) dz = \int_0^{2\pi} f(e^{it}) ie^{it} dt &= \int_0^{2\pi} \frac{\left(e^{it}\right)^2 + 1}{-ie^{it}\left(e^{it} - \alpha\right)\left(e^{it} - \beta\right)} ie^{it} dt \\ &= \int_0^{2\pi} \frac{\left(e^{it}\right)^2 + 1}{-\left(e^{it} - \alpha\right)\left(e^{it} - \beta\right)} dt = \int_0^{2\pi} \frac{\cos t}{a - \cos t} dt \end{aligned}$$

Now we can compute $\int_\gamma f(z)dz$ using the residue theorem. The pole at zero is inside γ. Since $a > 1$, α is outside the circle, and β is inside. The index of γ around both zero and β is one, so by the residue theorem

$$\int_\gamma f(z) dz = 2\pi i (\operatorname{res}_0 f + \operatorname{res}_\beta f)$$

We compute the residues by

$$\begin{aligned} \operatorname{res}_0 f &= \lim_{z \to 0} (z - 0) f(z) = \lim_{z \to 0} \frac{z^2 + 1}{-i(z - \alpha)(z - \beta)} = \frac{1}{-i\alpha\beta} = \frac{1}{-i} = i \\ \operatorname{res}_\beta f &= \lim_{z \to \beta} (z - \beta) f(z) = \lim_{z \to \beta} \frac{z^2 + 1}{-iz(z - \alpha)} = \frac{\beta^2 + 1}{-i\beta(\beta - \alpha)} = \frac{-ia}{\sqrt{a^2 - 1}} \end{aligned}$$

Thus

$$\int_0^{2\pi} \frac{\cos t}{a - \cos t} dt = \int_\gamma f(z) dz = 2\pi i \left(i - \frac{ia}{\sqrt{a^2 - 1}} \right) = 2\pi \left(\frac{a}{\sqrt{a^2 - 1}} - 1 \right)$$

□

NOTE: In the next problem, I switched the roles of a and b from the problem statement given by Prof Schenker in order to help myself emulate the example problem Example 3 from page 134 of Sarason and avoid confusing myself. Sorry if it confuses you.

Proposition 0.4 (Exercise 1). *Let $a > 1$ and $-1 < b < a - 1$. Then*

$$\int_0^\infty \frac{x^b}{1+x^a}dx = \frac{\pi}{a\sin\left(\frac{\pi(b+1)}{a}\right)}$$

Proof. We choose branches of z^a and z^b on the same slit plane, both taking value 1 at $z = 1$. Then the function

$$f(z) = \frac{z^b}{1+z^a}$$

is holomorphic on the slit plane except at a simple pole at $z_0 = e^{\pi i/a}$. For $r > 0$, let A_r be the circular arc $\left\{r^{i\theta} : 0 < \theta < \frac{2\pi}{a}\right\}$ oriented counterclockwise. For $0 < \epsilon < R$, let $\Gamma_{\epsilon,R}$ denote the closed curve

$$\Gamma_{\epsilon,R} = [\epsilon, R] \cup A_R \cup \left[Re^{2\pi i/a}, \epsilon e^{2\pi i/a}\right] \cup -A_\epsilon$$

Then $\Gamma_{\epsilon,R}$ winds once around z_0, so by the residue theorem,

$$\int_{\Gamma_{\epsilon,R}} f(z)dz = 2\pi i \operatorname{res}_{z_0} f$$

We can compute the residue since the denominator has a simple pole at z_0, by

$$\operatorname{res}_{z_0} \frac{z^b}{1+z^a} = \frac{z_0^b}{az_0^{a-1}} = \frac{e^{\pi ib/a}}{ae^{\pi i(a-1)/a}} = \frac{e^{\pi ib/a}}{ae^{\pi ia/a}e^{-\pi i/a}} = \frac{-e^{\pi ib/a}e^{\pi i/a}}{a} = \frac{-e^{\pi i(b+1)/a}}{a}$$

Thus

$$\int_{\Gamma_{\epsilon,R}} f(z)dz = \frac{-2\pi ie^{\pi i(b+1)/a}}{a}$$

Over the interval $[\epsilon, R]$, we have

$$\int_{[\epsilon,R]} f(z)dz = \int_\epsilon^R \frac{x^b}{1+x^a}dx$$

Over the segment $\left[\epsilon e^{2\pi i/a}, Re^{2\pi i/a}\right]$, we use the parametrization $\gamma(t) = te^{2\pi i/a}$ for $t \in [\epsilon, R]$, and get

$$\int_{[\epsilon e^{2\pi i/a},Re^{2\pi i/a}]} f(z)dz = \int_\epsilon^R \frac{(te^{2\pi i/a})^b}{1+(te^{2\pi i/a})^a}e^{2\pi i/a}dt = e^{2\pi i/a}\int_\epsilon^R \frac{t^be^{2\pi ib/a}}{1+t^ae^{2\pi ia/a}}dt$$
$$= e^{2\pi i/a}e^{2\pi ib/a}\int_\epsilon^R \frac{t^b}{1+t^a}dt = e^{2\pi i(b+1)/a}\int_\epsilon^R \frac{x^b}{1+x^a}dx$$

This says that

$$\left(1 - e^{2\pi i(b+1)/a}\right)\int_\epsilon^R \frac{x^b}{1+x^a}dx = \frac{-2\pi ie^{\pi i(b+1)/a}}{a} + \int_{A_\epsilon} f(z)dz - \int_{A_R} f(z)dz \tag{0.1}$$

We will show that the contributions from the integrals over A_ϵ and A_R are zero in the limit as $\epsilon \to 0$ and $R \to \infty$. Consider the integral over A_ϵ.

$$\int_{A_\epsilon} \frac{z^b}{1+z^a} dz = \int_0^{2\pi/a} \frac{(\epsilon e^{it})^b}{1+(\epsilon e^{it})^a} \epsilon i e^{it} dt = i \int_0^{2\pi/a} \frac{\epsilon^{b+1} e^{it(b+1)}}{1+\epsilon^a e^{ita}} dt$$

As $\epsilon \to 0$, the denominator of the integrand goes to one and the numerator goes to zero, so

$$\lim_{\epsilon \to 0} \int_{A_\epsilon} f(z) dz = 0$$

Now consider the integra over A_R.

$$\int_{A_R} \frac{z^b}{1+z^a} dz = \int_0^{2\pi/a} \frac{(Re^{it})^b}{1+(Re^{it})^a} Rie^{it} dt = i \int_0^{2\pi/a} \frac{R^{b+1} e^{it(b+1)}}{1+R^a e^{ita}} dt$$

We can estimate the integrand by

$$\left| \frac{R^{b+1} e^{it(b+1)}}{1+R^a e^{ita}} \right| = \frac{R^{b+1}}{R^a - 1}$$

By hypothesis, $b+1 < a$, so this tends to zero as $R \to \infty$. Now we can take the limit of Equation 0.1 as $\epsilon \to 0$ and $R \to \infty$, and the A_ϵ, A_R terms drop, so

$$\lim_{\epsilon \to 0} \lim_{R \to \infty} \left(1 - e^{2\pi i(b+1)/a}\right) \int_\epsilon^R \frac{x^b}{1+x^a} dx = \frac{-2\pi i e^{\pi i(b+1)/a}}{a}$$

which we can rewrite as

$$\int_0^\infty \frac{x^b}{1+x^a} dx = \frac{-2\pi i e^{\pi i(b+1)/a}}{a(1-e^{2\pi i(b+1)/a})}$$

Now we attempt to simplify this, so that we can convince ourselves that it's actually a real number. We make the substitution $t = \frac{\pi(b+1)}{a}$.

$$\begin{aligned} \frac{-2\pi i e^{\pi i(b+1)/a}}{a(1-e^{2\pi i(b+1)/a})} &= \left(\frac{-2\pi i}{a}\right) \frac{e^{it}}{1-e^{2it}} = \left(\frac{-2\pi i}{a}\right) \frac{e^{it}}{(1-e^{it})(1+e^{it})} = \left(\frac{-2\pi i}{a}\right) \left(\frac{i}{2\sin(t)}\right) \\ &= \frac{\pi}{a\sin(t)} = \frac{\pi}{a\sin\left(\frac{\pi(b+1)}{a}\right)} \end{aligned}$$

□

Proposition 0.5 (Exercise 2). *Let $k \in \mathbb{R}$. Then*

$$\int_{-\infty}^\infty \frac{e^{ikx}}{1+x^4} dx = \frac{\pi e^{-k\sqrt{2}/2}\Big(\cos(k\sqrt{2}/2) + \sin(k\sqrt{2}/2)\Big)}{\sqrt{2}}$$

Proof. Define

$$f(z) = \frac{e^{ikz}}{1+z^4}$$

Then f is holomorphic on $\mathbb{C}$ except for the fourth roots of -1, which are $e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}$, and $e^{\pi i/4}$. Note that the first two lie in the upper half plane and the second two lie in the lower half plane. Let $z_1 = e^{\pi i/4}$ and $z_2 = e^{3\pi i/4}$.

For $R > 0$, let S_R be the semicircle in the upper half plane parametrized by $S_R(t) = Re^{it}$ with $t \in [0, \pi]$. Let Γ_R be the contour $[-R, R] \cup S_R$. For $R > 1$, Γ_R winds once around z_1 and z_2 but not around the other roots, so by the Residue Theorem,

$$\int_{\Gamma_R} f(z)dz = 2\pi i(\operatorname{res}_{z_1} f + \operatorname{res}_{z_2} f)$$

We can compute the residues using Exericise VIII.12.1, since they are simple poles:

$$\operatorname{res}_{e^{\pi i/4}} \left(\frac{e^{ikz}}{1+z^4}\right) = \frac{e^{ike^{\pi i/4}}}{4\left(e^{\pi i/4}\right)^3} = \frac{e^{ike^{\pi i/4}}}{4e^{3i\pi/4}}$$

$$\operatorname{res}_{e^{3\pi i/4}} \left(\frac{e^{ikz}}{1+z^4}\right) = \frac{e^{ike^{3\pi i/4}}}{4\left(e^{3\pi i/4}\right)^3} = \frac{e^{ike^{3\pi i/4}}}{4e^{9\pi i/4}} = \frac{e^{ike^{3\pi i/4}}}{4e^{i\pi/4}}$$

We claim that the integral $\int_{S_R} f(z)dz$ goes to zero as R goes to infinity. We have the following chain of inequalities,

$$\left|\int_{S_R} f(z)dz\right| \leq \int_{S_R} |f(z)|dz = \int_0^\pi \frac{|e^{ikRe^{it}}|}{|1+(Re^{it})^4|}dt$$
$$= \int_0^\pi \frac{e^{-kR\sin t}}{|1+R^4e^{4it}|}dt \leq \int_0^\pi \frac{1}{R^4-1}dt \leq \frac{\pi R}{R^4-1}$$

And clearly the limit as $R \to \infty$ of the far right is zero, so the claim is proven. Thus

$$\lim_{R\to\infty} \int_{[-R,R]} f(z)dz = \lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 2\pi i(\operatorname{res}_{z_1} f + \operatorname{res}_{z_2} f)$$

which we rewrite as

$$\lim_{R\to\infty} \int_{-R}^R \frac{e^{ikz}}{1+z^4}dz = \int_{-\infty}^\infty \frac{e^{ikx}}{1+x^4}dx = 2\pi i \left(\frac{e^{ike^{\pi i/4}}}{4e^{3i\pi/4}} + \frac{e^{ike^{3\pi i/4}}}{4e^{i\pi/4}}\right)$$

Finally, we do a tedious calculation to find the sum of the two residues.

$$\frac{e^{ike^{\pi i/4}}}{4e^{3i\pi/4}} + \frac{e^{ike^{3\pi i/4}}}{4e^{i\pi/4}} = \frac{e^{ike^{i\pi/4}} + e^{i\pi/2}e^{ike^{3\pi i/4}}}{4e^{3\pi i/4}} = \frac{e^{ike^{i\pi/4}} + ie^{ike^{3\pi i/4}}}{4(-\sqrt{2}/2 + i\sqrt{2}/2)}$$
$$= \frac{e^{ik\sqrt{2}/2 - k\sqrt{2}/2} + ie^{-ik\sqrt{2}/2 - k\sqrt{2}/2}}{-2\sqrt{2} + i2\sqrt{2}} = \frac{e^{ik\sqrt{2}/2}e^{-k\sqrt{2}/2} + ie^{-ik\sqrt{2}/2}e^{-k\sqrt{2}/2}}{-2\sqrt{2}(1+i)}$$
$$= \frac{e^{-k\sqrt{2}/2}\left(e^{ik\sqrt{2}/2} + ie^{-ik\sqrt{2}/2}\right)}{-2\sqrt{2}(1+i)}$$

Now we work with the right part of the product in the numerator.

$$\begin{aligned} e^{ik\sqrt{2}/2} + ie^{-ik\sqrt{2}/2} &= \cos(k\sqrt{2}/2) + i\sin(k\sqrt{2}/2) + i\cos(-k\sqrt{2}/2) + i^2\sin(-k\sqrt{2}/2) \\ &= \cos(k\sqrt{2}/2) + i\sin(k\sqrt{2}/2) + i\cos(k\sqrt{2}/2) + \sin(k\sqrt{2}/2) \\ &= (1+i)\big(\cos(k\sqrt{2}/2) + \sin(k\sqrt{2}/2)\big) \end{aligned}$$

Note that $\frac{1+i}{1-i} = i$, so

$$\begin{aligned} \frac{e^{ike^{\pi i/4}}}{4e^{3i\pi/4}} + \frac{e^{ike^{3\pi i/4}}}{4e^{i\pi/4}} &= \frac{e^{-k\sqrt{2}/2}(1+i)\Big(\cos(k\sqrt{2}/2) + \sin(k\sqrt{2}/2)\Big)}{-2\sqrt{2}(1-i)} \\ &= \frac{ie^{-k\sqrt{2}/2}\Big(\cos(k\sqrt{2}/2) + \sin(k\sqrt{2}/2)\Big)}{-2\sqrt{2}} \end{aligned}$$

Thus

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^4}dx = 2\pi i\left(\frac{e^{ike^{\pi i/4}}}{4e^{3i\pi/4}} + \frac{e^{ike^{3\pi i/4}}}{4e^{i\pi/4}}\right) &= 2\pi i\left(\frac{ie^{-k\sqrt{2}/2}\Big(\cos(k\sqrt{2}/2) + \sin(k\sqrt{2}/2)\Big)}{-2\sqrt{2}}\right) \\ &= \frac{\pi e^{-k\sqrt{2}/2}\Big(\cos(k\sqrt{2}/2) + \sin(k\sqrt{2}/2)\Big)}{\sqrt{2}} \end{aligned}$$

(Aside: We shouldn't be surprised that this integral is real, because even though the function isn't real valued on the real axis, the imaginary part is

$$\frac{\sin(kx)}{1+x^4}$$

which is an odd function, so the symmetric integral

$$\int_{-R}^{R} \frac{\sin(kx)}{1+x^4}dx$$

must be zero. End aside.) □